OBJECTIVES

Simple mechanical oscillators are systems which, when slightly disturbed from their equilibrium, will oscillate with harmonic motion. This is the most elemental vibration of a single particle or a one-dimensional system. Simple mechanical oscillators involve a mass $m$, which has inertia, and a spring with stiffness $K$, which provides a restoring force. When the forces imparted by the mass and spring can be made to cancel, the total energy of the mechanical system can be large, resulting in large amplitude oscillations or resonance. In practice mechanical oscillators are damped, and their energy is dissipated by a resistive or viscous process, which bounds the amplitude of the oscillation, even at resonance. In addition, mechanical oscillators may be driven by external forces, which supply energy to the system. The goal of this experiment is to study the behavior of these forced harmonic oscillators near resonance.

The mechanical oscillator follows the same form of differential equation of motion as the current in an electrically resonant circuit. Due to this similarity, and because electromechanical transducers are used in this experiment to excite and to measure the response of a mechanical system, the experiment introduces the principle of transduction -- the conversion of energy from one system into energy of another. In this case, one transducer will be used to convert electrical energy into a mechanical driving force (the attraction between a magnet and an electromagnet) and the other will be used to convert mechanical vibrations into an electrical signal (a fiber-optic probe). The concept of impedance is key for understanding electromechanical transduction. Mechanical impedance is defined as the ratio of the mechanical driving force to the resulting velocity. A material with high impedance will require large driving forces to produce even small vibration velocities. A material with low impedance will respond with high velocity vibrations when even a small driving force is applied.

THEORY

SIMPLE MECHANICAL OSCILLATOR: A mass is a mechanical component whose inertial force/displacement relationship is governed by Newton’s law:

$$F = - m \ a = - m \{d^2x/dt^2\},$$

where $m$ is the mass and $a$ is the acceleration. A spring is a component obeying the Hooke’s force/displacement relationship:

$$F = - K \ x,$$
where $K$ is the "spring constant", and $x$ is the change in the length of the spring from its rest position. A mechanical resistance provides a force proportional to the velocity,

$$F = -R \frac{dx}{dt}.$$

The simple mechanical oscillator is a combination of these three components as shown in Figure 2-1. One end of the spring is fixed and the other is coupled to the mass.

![Figure 2-1 Forced mechanical oscillator with force $F$, applied to mass $m$, which is connected to spring $k$, and viscous damping $R$.](image)

The displacement $x(t)$ is the mass offset from the rest position as a function of time. $F(t)$ is an external exciting force applied to the mass. The motion of the mass will be described by the differential equation balancing the forces,

$$F(t) = m \left\{ \frac{d^2 x}{dt^2} \right\} + R \left\{ \frac{dx(t)}{dt} \right\} + K x(t)$$

Assuming a sinusoidal driving force $F_0 \exp(-i\omega t)$ and a sinusoidal displacement $x(t) = x_0 \exp(-i\omega t)$, we obtain:

$$F_0 = -\omega^2 m x_0 - i\omega R x_0 + K x_0$$

The mechanical impedance $Z$ is defined as the ratio of force to velocity. Thus,

$$Z = \frac{F}{\text{Vel}} = F \left\{ \frac{dx}{dt} \right\}$$

$$Z = \{(K - \omega^2 m -iR\omega) / \{-i\omega\} = (-i\omega m + iK/\omega) + R$$

where $R$ is the mechanical damping. For the spring-mass system, the Quality factor, $Q$, is a measure of the energy stored divided by the energy lost in a single cycle of oscillation. One measure of $Q$ is:

$$Q = \{m K\}^{1/2} / R$$
which is a dimensionless quantity. Another means to estimate \( Q \) is from the ratio of the bandwidth, \( BW \) (2-sided bandwidth between 3dB points relative to the peak), to the resonant frequency \( f_0 \) as follows:

\[
Q = \frac{f_0}{BW}
\]

(for example, a 100 Hz resonant frequency and 10 Hz bandwidth system has a \( Q \) of 10).

The impedance, \( Z \), is a minimum at the resonant frequency, \( \omega_c = \sqrt{\frac{k}{m}} \) and the stored energy, \( E = -\omega m + k/\omega = 0 \). At this frequency the mechanical reactance, \( E \) goes to zero.

EXPERIMENTAL SETUP

We will fabricate a mechanical oscillator using a thin beam of stainless-steel as a “leaf spring” and a magnet as the mass. The driving force will be supplied by the attraction between the magnetic mass and an electromagnet, made using a coil with a steel (high permittivity) core. Figure 2 illustrates the mechanical oscillator and its components:

![Figure 2](image)

Figure 2. Schematic illustration of the mechanical oscillator and measurement electronics.

A mass (magnet) is attached to one end of a leaf spring, which is in turn supported by the laser-cut box on the other end. A coil attached to the base-plate of the box, has a
high permittivity core (steel screw) to create an electromagnet. A sinusoidal current in the coil creates a force by interaction between the magnetic field of the mass, and the magnetic field of the coil. The mass motion is sensed by a fiber-optic probe, whose output is proportional to the displacement of the mass/spring. As the force is applied by the coil the mass will oscillate about its equilibrium position. The oscillation will be damped due to energy required to move air, losses in the spring, and in the magnetic circuit. Determining the response of the mass/spring, and the mechanical impedance, that is, the relation between the driving force applied by the coil and velocity of the mass/spring is a primary goal of our experiment.

An oscillating current creates an harmonic magnetic field from the coil. For varying frequencies, if the current supplied to the coil is kept constant, the force applied on the mass/spring will be constant.

To estimate the motion of the mass/spring, we use a fiber-optic probe. This device consists of a bundle of light emitting and detecting optical fibers. Two sets of optical fibers are joined into a single bundle. Half of the fibers emit light, while half of the fibers detect reflected light. The level of reflected light is related to the distance between the probe and the reflective surface. Calibration of the fiber-optic probe is accomplished with a micrometer. As the micrometer moves closer to the probe by a known distance, the output of the probe electronics is recorded. The result is a measurement of the calibration coefficient $k_{fop}$ (distance/voltage). The probe output voltage $V_{fop}$ multiplied by the calibration coefficient gives the displacement of the mass. Multiplying by $-i\omega$ yields the velocity.

To estimate the spring constant a small known mass (e.g. $m_{test} = 10 \text{ g}$) is attached at the location of the mass/magnet and this displacement is observed with the fiber optic probe. The spring constant is calculated from the ratio of the applied force ($F = 9.8 \text{ m/s}^2 \times m_{test}$) to the displacement of the mass ($x = V_{fop} \times K_{fop}$).

$$K_{spring} = \frac{(9.8 \text{ m/s}^2 \times m_{test})}{(V_{fop} \times K_{fop})}$$

Note: for the leaf spring $K_{spring} = \sim 150$ to 190 N/m, depending on the exact length of the spring.

Measurement of the response of the mechanical oscillator will involve making amplitude and phase estimates both for the driving force (current in the coil) and the mass velocity as measured by the fiber optic probe. Both pairs of measurements must be made with the same phase reference. The actual phase reference can be completely arbitrary as long as it is the same for both measurements. We can express the amplitude and phase as components of complex voltages. Thus, the fiber optic probe voltage $V_{fop}$ is represented by:

$$V_{fop} = a \exp(-i\omega t+ib)$$
whose standard deviation is given by \( \delta V_{\text{top}} = \sqrt{\delta a^2 + a^2 \delta b^2} \), where \( a \) is the amplitude and \( b \) is the phase of the voltage measurement and \( \delta a \) and \( \delta b \) are their standard deviations.

The probe voltage is related to the displacement \( D \), by the calibration coefficient \( k_{\text{top}} \) as follows:

\[
D_{\text{top}} = k_{\text{top}} a \exp(-i\omega t + ib)
\]

and taking the derivative gives the velocity:

\[
V_{\text{ELtop}} = -i\omega k_{\text{top}} a \exp(-i\omega t + ib)
\]

The driving force \( F \) is derived from the current in the coil. We measure the voltage across the coil as:

\[
V_{\text{coil}} = c \exp(-i\omega t + id)
\]

and the standard deviation is given by \( \delta V_{\text{geo}} = \sqrt{\delta c^2 + c^2 \delta d^2} \), where \( c \) is the amplitude and \( d \) is the phase of the voltage measurement and \( \delta c \) and \( \delta d \) are the standard deviations.

We assume a fixed impedance for the coil \( (R_{\text{coil}} = \sim 20 \text{ ohms}) \). This is only true for low frequencies, as for our measurements \((< 200 \text{ Hz})\), and at higher frequencies the coil impedance will increase as \( i\omega L \), where \( L \) is the inductance of the coil. The current in the coil is derived from the measured voltage \( V_{\text{coil}} \) divided by the resistance:

\[
I_{\text{coil}} = (1/R_{\text{coil}}) c \exp(-i\omega t + id)
\]

The force applied to the mass can be calculated from product of the current and the force constant \( K_{\text{force}} \) \((\text{N/amp})\), where we obtain the force constant at low frequency \((\text{below resonance})\):

\[
K_{\text{force}} = (K_{\text{spring}} x |D_{\text{top}}|) / |I_{\text{coil}}|
\]

and the \( | | \) symbols denote the absolute value. For the coil system in this lab the force constant is about \( K_{\text{force}} = 0.1\text{N/amp} \).

The mechanical impedance is, in turn, the driving force \( F \), divided by the velocity of the suspended mass:

\[
Z = F / V_{\text{ELtop}}
\]
\[ Z = k_{\text{force}} \cdot l_{\text{coil}} / \text{VEL}_{\text{top}} \]

where the time variation has been cancelled by taking the ratio.

**DATA COLLECTION**

**DISPLACEMENT PROBE CALIBRATION:** The static calibration of the fiber optic probe is done using a micrometer to vary the distance of the probe from the reflective surface on the leaf spring, with the mass stationary. The dc output voltage is measured as a function of changes in the micrometer setting. The micrometer is calibrated in metric units, (.01 millimeters per small division). Find the linear output voltage range and fit a straight line to the data over this range.

After calibrating the fiber optic probe, position it so that the output voltage is in the linear range of the probe electronics (e.g. ~6 volts). Note for the fiber optic probe the calibration is approximately: \( K_{\text{fob}} = 0.4 \text{ mm/v} \)

**CALIBRATION OF SPRING CONSTANT:** To estimate the spring constant a small known mass (weight) is attached at the location of the mass/magnet and this displacement is observed with the fiber optic probe. The spring constant is calculated from the ratio of the applied force to the displacement of the mass. It may be necessary to position the probe at a distance less than the location of the magnetic mass. In this case increase the measured displacement by the ratio of the probe location (e.g. 6 cm) and the location of the applied weight (e.g. 7 cm).

**DRIVER COIL CALIBRATION:** Apply power to the coil from the oscillator. Starting at 10 Hz, sweep the frequency slowly upward to locate the very narrow response at the resonant frequency (should be near 35 Hz). Then set the oscillator at about 1/10 of the resonant frequency (~3Hz) and adjust the current to a peak to peak value of 50 mA ampere (50 mV volt across the series 1 ohm resistor). Measure the peak-to-peak probe voltage to obtain a displacement/ampere ratio. Convert this by use of the measured spring constant to obtain a force/ampere ratio, \( k_{\text{force}} \). If the system noise is too high to permit accurate measurements, the oscilloscope can be put into an averaging mode for interference reduction.

**OSCILLATOR MASS:** The effective mass of the oscillator should be calculated from the sum of the mass of the magnet and an approximate contribution of the leaf spring. Use 1/3 of the mass of the leaf spring as the effective mass projected out to the center of gravity of the mass/magnet.

**SIMPLE OSCILLATOR RESPONSE DATA:**
In all of the response measurements be careful to keep the drive level below any nonlinear region of operation. Distortion of the waveform is the best indication of distortion in the electrical system. Measure the mass displacement and driving voltage
as a function of frequency over a frequency range of at least 0.3 to 3 times the resonant frequency. Before collecting data, manually set the oscillator to the upper frequency limit and then to the lower frequency limit and observe the waveform on the oscilloscope to be sure that the waveform is not distorted from overdriving. Collect data for a fully extended leaf spring and for an extension of about ½ its length.

Frequency response measurements are carried out using LABVIEW to control the signal generator and oscilloscope. Measurement of both amplitude and phase is used to obtain the complex coefficients. If the waveform generator is in the normal (continuous) mode and the oscilloscope trigger is derived from the sync out of the waveform generator, the phase relationship between the waveform generator and the oscilloscope sampling interval will remain constant so the fiber optic probe and driving current samples can be measured sequentially.

In all of the response measurements be careful to keep the drive level below any nonlinear region of operation. Distortion of the waveform is the best indication of distortion in the electrical system.

ANALYSIS

Use the fiber optic probe data to characterize the velocity of the mass \((-i\omega V_{fop} k_{fop})\), plot the response versus frequency.

Plot the experimentally measured complex impedance, \(Z\).

Derive a value for the spring constant \(K_{\text{spring}}\) both from direct measurement and from use of the resonance frequency.

Estimate the damping \(R\) to best match the experimental impedances.

From the damping \(R\), estimate the \(Q\) of the system. Likewise estimate the \(Q\) from the bandwidth of the resonance.
Design of the Mechanical Oscillator:

Use ¼ inch birch-wood and cut the components of the box-framework for the mechanical oscillator using a laser cutter. The design is appended below:

A leaf spring can be made from stainless steel about 10 cm x 1 cm x .1-.2 cm. ¾ to 1 inch wood screws hold the box together.
The electro-magnet coil is obtained from a JF-0630B open frame solenoid, with the core replaced by a steel bolt. The mass is a neodymium disk magnet.