EXPERIMENT 11 - Normal Modes in a Channel

OBJECTIVES

To study wave propagation in a fluid filled channel with pressure release boundaries, and to compare the predictions of normal mode theory with measurements.

THEORY

Normal mode theory is one of the ways that wave propagation in a bounded medium may be described. In the ocean the surface of the sea and the seafloor are the two major boundaries that influence the propagation. One of these, the sea surface is a nearly perfect reflector, the other, the seafloor, is partially reflecting with a reflectivity that varies from place to place. In the laboratory we will study a simpler case to illustrate propagation in a waveguide. Here, a rectangular channel, bounded by styrofoam on three sides and air on the fourth, provides simple, pressure release boundary conditions which are readily analyzed theoretically. The styrofoam, by virtue of its low mass and high compressibility has a low acoustic impedance compared to water, as does the air. For the purpose of theoretical analysis the boundary impedance from both styrofoam and air can be set to zero to a good approximation.

The wave propagation in the water is governed by the wave equation:

\[ \nabla^2 p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} \]  \quad (Eq 1)

In the appropriate cartesian coordinate system,

\[ \nabla^2 p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} \]  \quad (Eq 2)

assuming a solution of the form,

\[ p = \exp(ax + by + dz - i\omega t) \]  \quad (Eq 3)

Substituting eq. 3 into eq. 2, the relationship between the coefficients is given by,

\[ a^2 + b^2 + d^2 = \frac{\omega^2}{c^2} \]  \quad (Eq 4)

which yields roots of \( \pm a, \pm b, \pm d \).
The boundary conditions to be met, letting $x$ be along the axis of the channel and unbounded, $y$ in the horizontal, across the axis of the channel, and $z$ in the vertical, are:

$$ p = 0 \quad \text{at} \quad y = 0, W \quad \text{(Eq 5)} $$

$$ p = 0 \quad \text{at} \quad z = 0, H $$

A general solution would then be

$$ p = (Ae^{ax} + A'e^{-ax})(Be^{by} + B'e^{-by})(De^{dz} + D'e^{-dz})e^{-i\omega t} \quad \text{(Eq 6)} $$

At $y = 0$, $B + B' = 0$, $B = -B'$ and at $y = W$, $\exp(bW) - \exp(-bW) = 0$

which is only satisfied if $b$ is imaginary i.e., $b = ik_y$ such that

$$ \exp(ik_yW) - \exp(-ik_yW) = 2i \sin(k_yW) = 0 $$

which is true for certain values of $k_y$

$$ k_y = \frac{n\pi}{W}, \quad n = 1, 2, 3, ... $$

Similarly we have

$$ k_z = \frac{m\pi}{H}, \quad m = 1, 2, 3, ... $$

Solving for $a$,

$$ a^2 = -b^2 - d^2 - \frac{\omega^2}{c^2} $$

$$ a = \pm \left[ \left( \frac{n\pi}{W} \right)^2 + \left( \frac{m\pi}{H} \right)^2 - \left( \frac{\omega}{c} \right)^2 \right]^{1/2} \quad \text{(Eq 7)} $$

If,

$$ \left( \frac{n\pi}{W} \right)^2 + \left( \frac{m\pi}{H} \right)^2 < \left( \frac{\omega^2}{c^2} \right) $$

then $a$ is imaginary, i.e., $a = ik_x$

$$ k_x = \pm \left[ \left( \frac{\omega}{c} \right)^2 - \left( \frac{n\pi}{W} \right)^2 - \left( \frac{m\pi}{H} \right)^2 \right]^{1/2} \quad \text{(Eq 8)} $$

and the solution will be a traveling wave in the $x$ direction with a phase velocity
\[ c_x = \frac{\omega_x}{k_x} = c \left[ 1 - \left( \frac{nc\pi}{\omega W} \right)^2 - \left( \frac{mc\pi}{\omega H} \right)^2 \right]^{-1/2} \]  \hspace{1cm} \text{(Eq 9)}

and \( p \) will have the form

\[ p_{nm} = \sin \left( \frac{n\pi y}{W} \right) \sin \left( \frac{m\pi z}{H} \right) e^{i(\pm k_x x - \omega t)} \]  \hspace{1cm} \text{(Eq 10)}

This is the \( (n,m)_m \) normal mode. The zero pressure boundary conditions make the amplitude of the zeroth order mode zero, so that, for propagation to take place, both \( n \) and \( m \) must be greater than zero.

The cutoff frequency for the \( n,m \) mode is where,

\[ (2\pi f)^2 = \omega^2 = \left[ \left( \frac{n\pi}{W} \right)^2 + \left( \frac{m\pi}{H} \right)^2 \right] c^2 \]  \hspace{1cm} \text{(Eq 11)}

If \( a \) is real,

\[ \left( \frac{n\pi}{W} \right)^2 + \left( \frac{m\pi}{H} \right)^2 > \left( \frac{\omega^2}{c^2} \right) \quad \text{then} \quad a = \alpha \]  \hspace{1cm} \text{(Eq 12)}

where

\[ \alpha = \left[ \left( \frac{n\pi}{W} \right)^2 + \left( \frac{m\pi}{H} \right)^2 - \left( \frac{\omega}{c} \right)^2 \right]^{1/2} \]  \hspace{1cm} \text{(Eq 13)}

and \( p \) will have the form

\[ p_{nm} = \sin \left( \frac{n\pi y}{W} \right) \sin \left( \frac{m\pi z}{H} \right) e^{\pm \alpha x} e^{-i\omega t} \]  \hspace{1cm} \text{(Eq 14)}

This motion represents a standing wave decaying exponentially. The positive and negative values of \( \alpha \) represent observation points to the left or right of the source location at \( x = 0 \).
PROCEDURE NOTES

Figure 1 illustrates the experimental setup. The channel has a width of 5 cm and a height of 2.5 cm. It is fitted with a pair of source transducers at one end and an absorbing wedge termination at the other end. A moveable probe is used to explore the phase and amplitude of the acoustic field in the channel. Put the channel in one of the aluminum troughs equipped with a cart. Mount the probe in the surface gauge fixture and tape it to the cart. Also tape the transducers in place so that they do not rotate; the transducers are not perfectly omnidirectional.

Electrical pickup between the source transducers and the receiving probe is particularly troublesome in this experiment because measurements are being made with steady state signals rather than with pulses. Good shielding practices, which sometimes seem to take on the nature of a black art, must be followed for good results. The following steps are guidelines.

1. In general, all instruments should be grounded separately to a common point -- in this case the oscilloscope chassis ground is a good choice.

2. Circuit grounds should be returned to that point without forming a closed loop (ground loop).

3. Outer electrodes of the transducers (shield braid on the cable) should be on the circuit ground or "low" side.

4. The receiving probe shield braid should be tied to the ground side of the preamplifier input and nowhere else -- the ground side of the preamp to the bandpass filter ground, bandpass filter ground to scope ground, and if all goes well, what you observe will be the acoustic pressure field not the stray electrical field.

5. A large part of the "black art" of shielding is distinguishing electrical pickup from the desired signal so that a cut-and-try method of grounding rearrangements can be carried out. How do you tell the difference?

After you are convinced that electrical pickup has been put in its place, proceed with the experiment.

Measure $W$ and $H$ and then compute the theoretical cutoff frequencies for the 1,1; 2,1; 1,2; and 2,2 modes. Use these as an aid for evaluating the propagation regimes you will measure.

Excitation of the 1,1 normal mode is enhanced by driving both transducers in phase at the same amplitude. Excitation of the 2,1 normal mode is enhanced by driving them out of phase. Two separate adjustable-gain drivers are provided to balance the levels out of the
two transducers. The levels can be checked by driving each one independently, and then measuring the received amplitude on the probe located in the center of the channel at some reference point.

Several propagation regimes can be observed:

A. Standing waves below the cutoff frequency for the 1,1 mode.

B. Pure 1,1 mode propagation above the 1,1 cutoff frequency and below the 2,1 cutoff.

C. Pure 2,1 mode propagation above 2,1 cutoff by carefully balancing the transducer drive with opposite polarity driving voltages to null out the 1,1 mode.

D. Mixed mode propagation evidenced by beats along the \ direction caused by the different phase velocities of the two modes. This will be clearest if the frequency is above the 1,1 and 2,1 cutoff but below either 2,2 or 1,2 cutoff.

ANALYSIS

Compare measured phase velocities and amplitudes with theoretical values for the above cases, including the imaginary phase velocity or attenuation of case A for a standing wave.
Figure 1 Block diagram for experiment.